H

THE COSINE RULE

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The **cosine rule** relates the three sides of a triangle and one of its angles.

Consider triangle ABC with side lengths a, b, and c as shown.

H THE COSINE RULE

Using Pythagoras' theorem in $\triangle$BCN,

$$a^2 = h^2 + (c-x)^2$$

$$\therefore \quad a^2 = h^2 + c^2 - 2cx + x^2$$

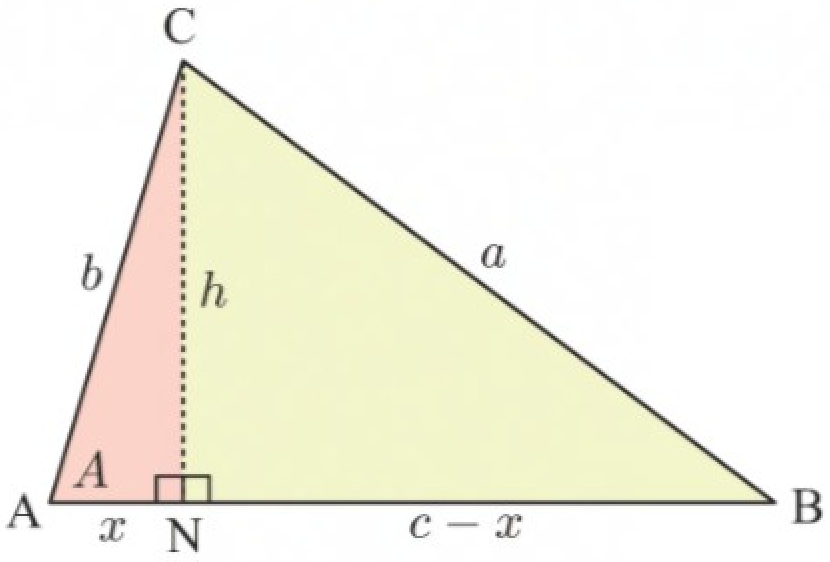
In $\triangle$ACN, $b^2 = h^2 + x^2$ and so $h^2 = b^2 - x^2$

Thus, $a^2 = (b^2 - x^2) + c^2 - 2cx + x^2$

$$\therefore \quad a^2 = b^2 + c^2 - 2cx$$

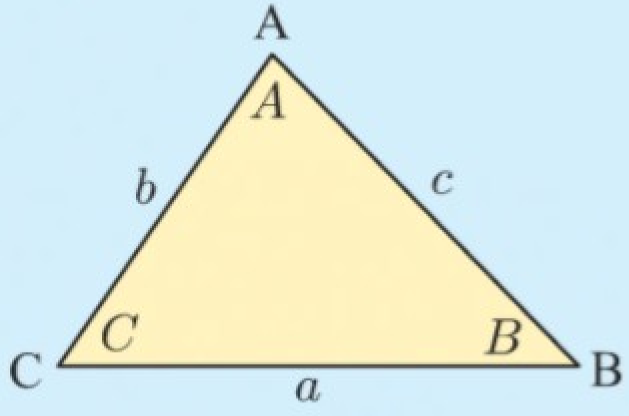
In $\triangle$ACN, $\cos A = \dfrac{x}{b}$ and so $x = b\cos A$

$$\therefore \quad a^2 = b^2 + c^2 - 2bc\cos A$$



In any triangle ABC with sides a, b, and c units, and opposite angles A, B, and C respectively,

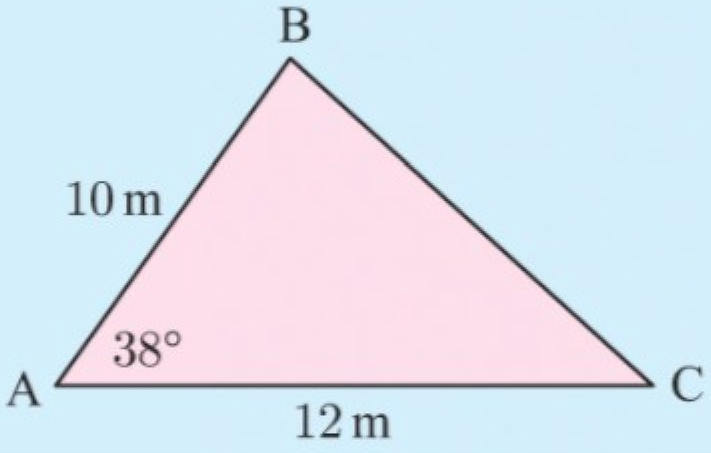
$$\begin{aligned} \boldsymbol{a^2} &= \boldsymbol{b^2 + c^2 - 2bc\cos A} \\ \boldsymbol{b^2} &= \boldsymbol{a^2 + c^2 - 2ac\cos B} \\ \boldsymbol{c^2} &= \boldsymbol{a^2 + b^2 - 2ab\cos C}. \end{aligned}$$



Example 16

Self Tutor

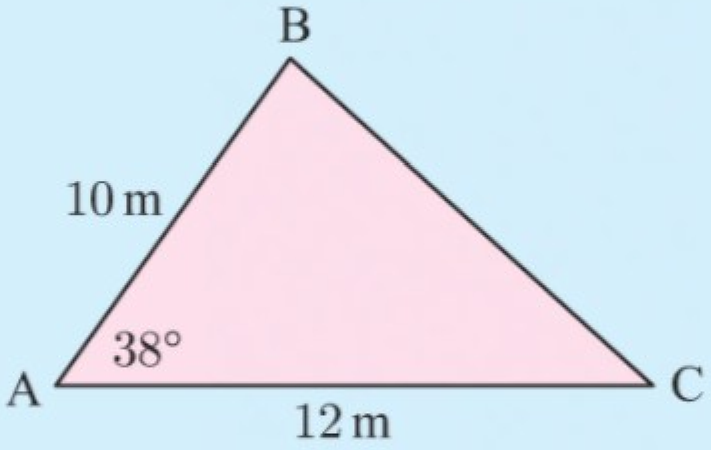
Find, correct to 2 decimal places, the length of [BC].

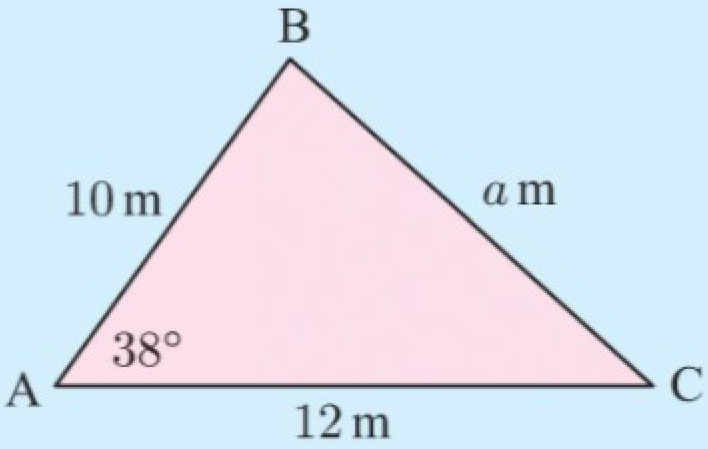


Example 16

Self Tutor

Find, correct to 2 decimal places, the length of [BC].





Using the cosine rule:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\therefore \quad a = \sqrt{12^2 + 10^2 - 2 \times 12 \times 10 \times \cos 38^\circ}$$

$$\therefore \quad a \approx 7.41$$

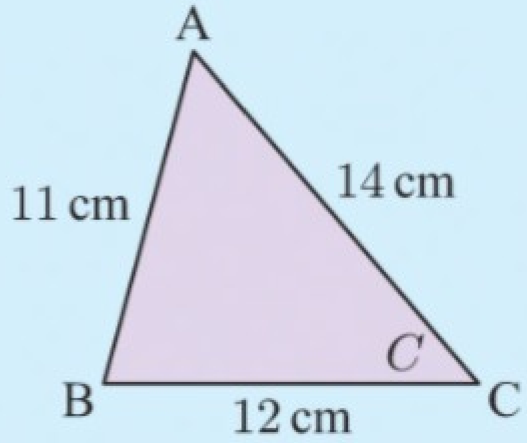
$\therefore$ [BC] is approximately 7.41 m long.

If we know all three side lengths of a triangle, we can use the cosine rule to find any of the angles. To do this, we rearrange the original cosine rule formulae:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac}, \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

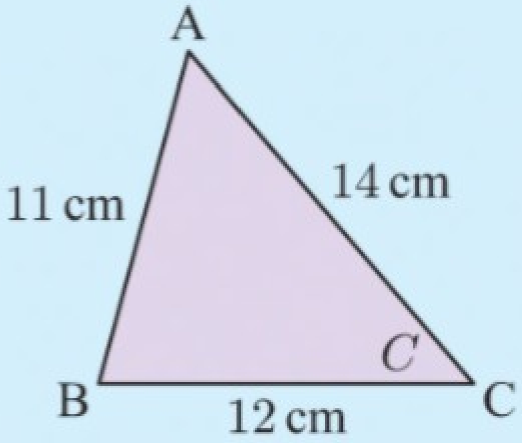
Example 17

Find the measure of angle C in the given figure.



Example 17

Find the measure of angle C in the given figure.



Self Tutor

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

$$\therefore \quad \cos C = \frac{12^2+14^2-11^2}{2\times 12\times 14}$$

$$\therefore \quad \cos C = \frac{219}{336}$$

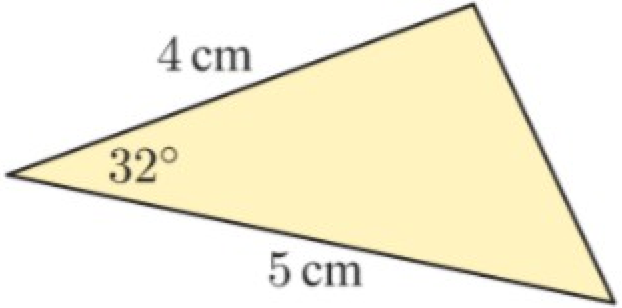
$$\therefore \quad C = \cos^{-1}\left(\frac{219}{336}\right)$$

$$\therefore \quad C \approx 49.3^\circ$$

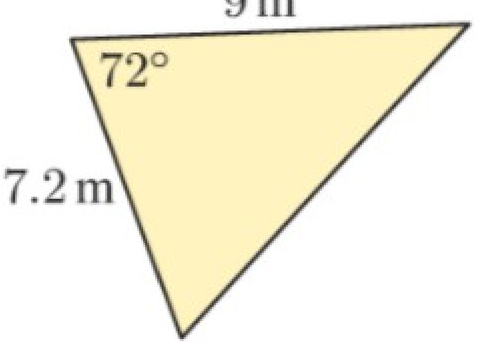
EXERCISE 12H

1 Find the length of the remaining side in the triangle:

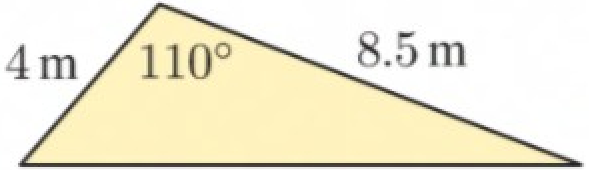
a



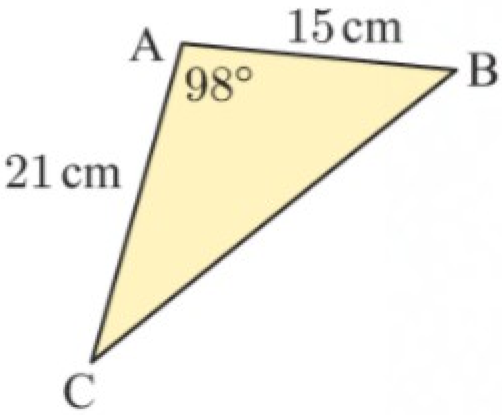
b



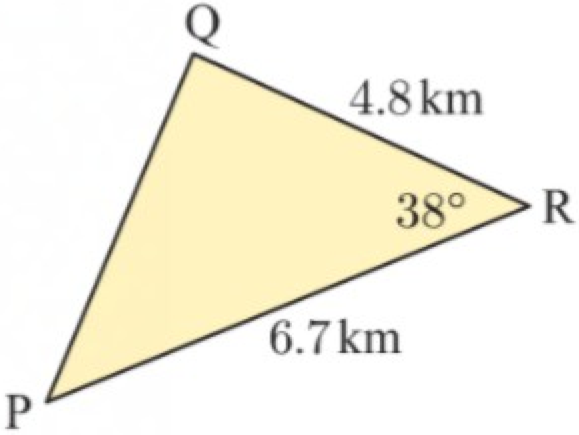
c



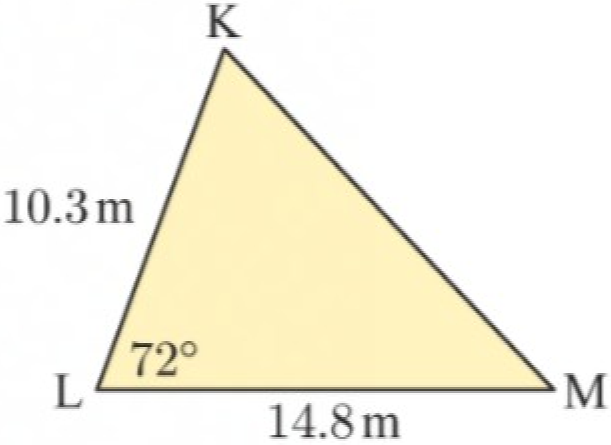
d



e

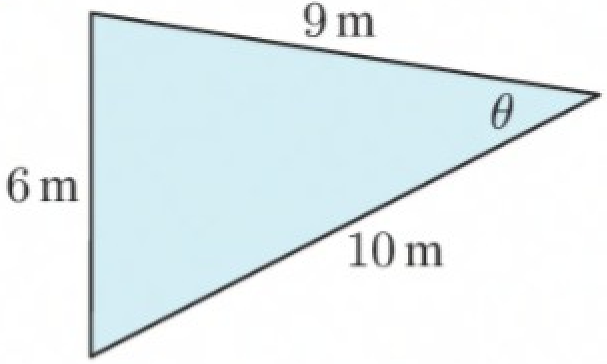


f



2 Find θ, rounded to 1 decimal place:

a



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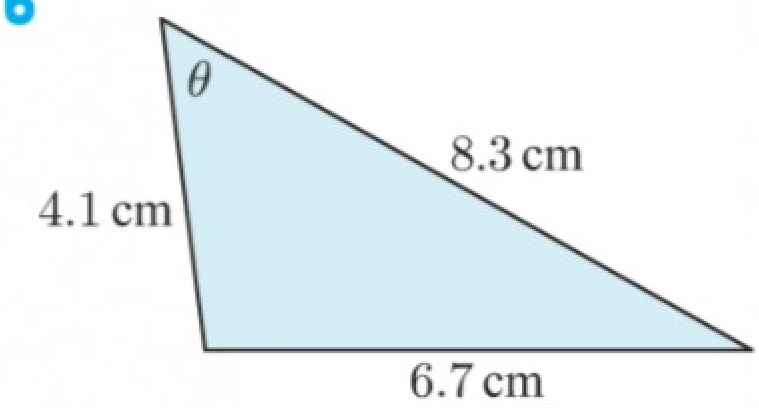

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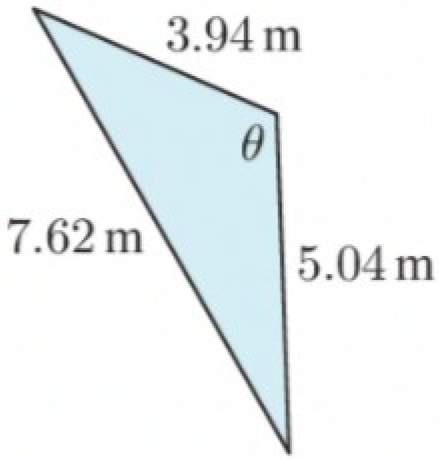
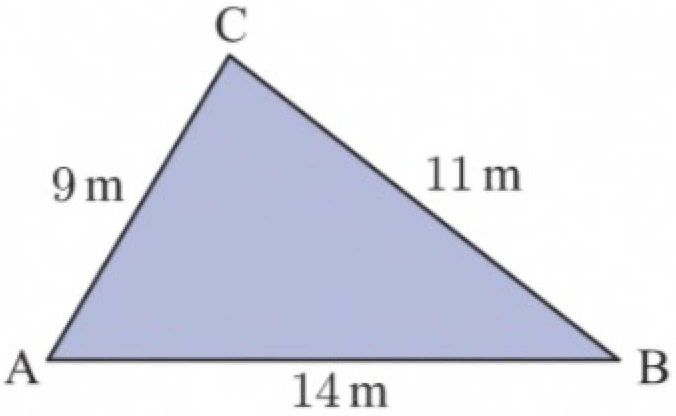
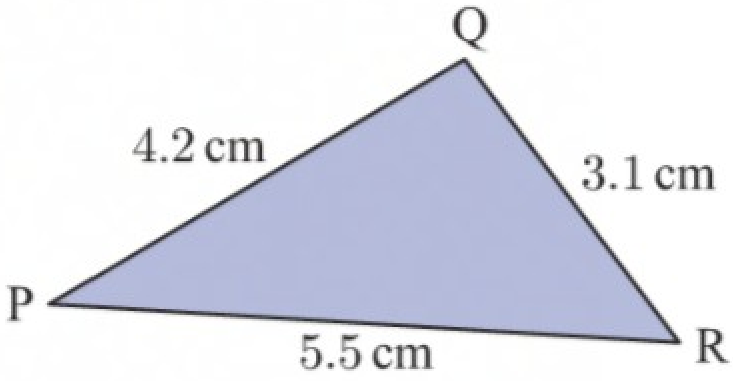


3 Find the measure of all angles of the triangle:

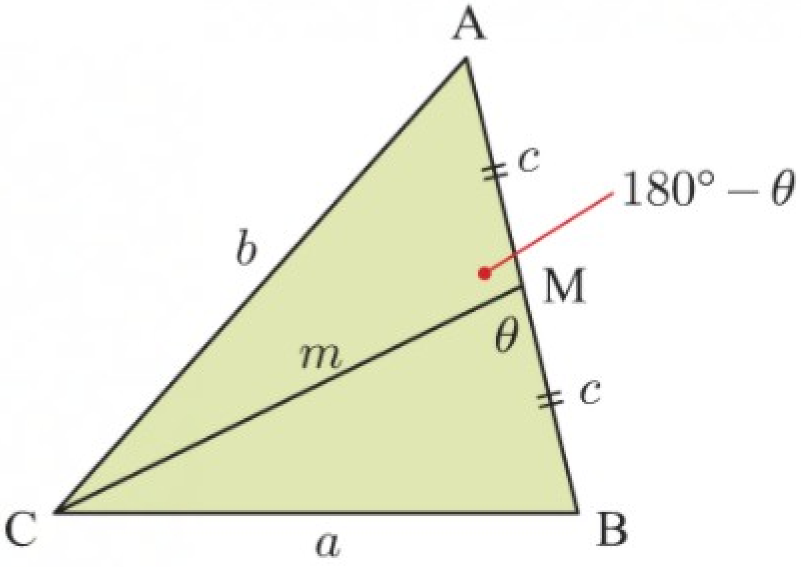
a



b



4



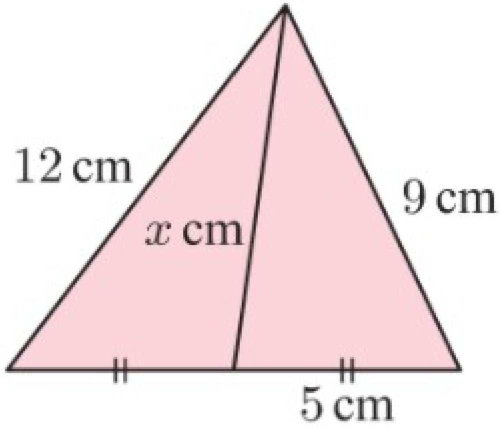
a Use the cosine rule in $\triangle$BCM to find $\cos\theta$ in terms of a, c, and m.

b Use the cosine rule in $\triangle$ACM to find $\cos(180° - \theta)$ in terms of b, c, and m.

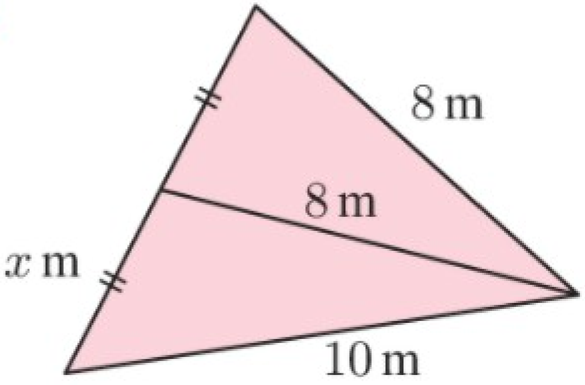
c Use the fact that $\cos(180° - \theta) = -\cos\theta$ to prove **Apollonius' median theorem**:
$a^2 + b^2 = 2m^2 + 2c^2$.

d Find x in the following:

i



ii



5 In triangle ABC, $AB = 10$ cm, $AC = 9$ cm, and $A\widehat{B}C = 60°$. Let $BC = x$ cm.

a Use the cosine rule to show that x is a solution of $x^2 - 10x + 19 = 0$.

b Solve this equation for x.

c Use a scale diagram and a compass to explain why there are two possible values of x.

EXERCISE 12H

1 **a** ≈ 2.66 cm **b** ≈ 9.63 m **c** 10.6 m
d 27.5 cm **e** 4.15 km **f** 15.2 m

2 **a** $\theta \approx 36.3°$ **b** $\theta \approx 53.2°$ **c** $\theta \approx 115.6°$

3 **a** $\widehat{\mathrm{A}} \approx 51.8°$, $\widehat{\mathrm{B}} \approx 40.0°$, $\widehat{\mathrm{C}} \approx 88.3°$
b $\widehat{\mathrm{P}} \approx 34.0°$, $\widehat{\mathrm{Q}} \approx 96.6°$, $\widehat{\mathrm{R}} \approx 49.3°$

4 **a** $\cos\theta = \dfrac{m^2 + c^2 - a^2}{2cm}$
b $\cos(180° - \theta) = \dfrac{m^2 + c^2 - b^2}{2cm}$
c **Hint:** $\cos(180° - \theta) = -\cos\theta$
d **i** $x \approx 9.35$ **ii** $x \approx 4.24$

5 **b** $x = 5 \pm \sqrt{6}$
c